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Difference equations for the co-recursive rth associated orthogonal polynomials of the D_q-Laguerre–Hahn class

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Abstract

We use some relations between the rth associated orthogonal polynomials of the $\mathscr{D}_q$-Laguerre–Hahn class to derive the fourth-order q-difference equation satisfied by the co-recursive rth associated orthogonal polynomials of the D_q-Laguerre–Hahn class.

When $r=1$ and for q-semi-classical situations, this q-difference equation factorizes as product of two second-order q-difference equations. Finally, we study some classical situations, and give some examples relative to the co-recursive associated discrete q-Hermite II orthogonal polynomials.

Keywords: Associated orthogonal polynomials; Co-recursive orthogonal polynomials; Laguerre–Hahn class

1. Introduction

Let $\mathscr{U}$ be a regular linear functional on the linear space $\mathscr{P}$ of the polynomials of real variable and $(P_n)_n$ the sequence of monic polynomials orthogonal with respect to $\mathscr{U}$ (see [7] for more details). As any standard orthogonal polynomial family, $(P_n)_n$ satisfies a three-terms recurrence relation

$$P_{n+1} = (x-\beta_n)P_n - \gamma_n P_{n-1}, \quad n \geqslant 1, \quad P_0 = 1, \quad P_1 = x - \beta_0, \tag{1}$$

where β_n and γ_n are complex numbers with $\gamma_n \neq 0$ $\forall n$. We assume that the linear functionals used in this paper are normalized by: $\langle \mathscr{U}, P_0^2 \rangle = \gamma_0 = 1$.

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- $\mathscr{U}$ and the corresponding monic orthogonal polynomials family are said to be of the $\mathscr{D}_q$-Laguerre–Hahn class if the Stieltjes function $S(\mathscr{U})$ of $\mathscr{U}$ satisfies a $\mathscr{D}_q$-Riccati q-difference equation

$$\phi(qx)\mathscr{D}_q S(x) = G(x,q)S(x)\mathscr{G}_q S(x) + E(x,q)S(x) + F(x,q)\mathscr{G}_q S(x) + H(x,q), \tag{2}$$

where $\phi \neq 0, G, E, F, G$ and H are polynomials in the variable x and the operators $\mathscr{D}_q$ and $\mathscr{G}_q$ are defined by

$$\mathscr{D}_q P(x) = \frac{P(qx)-P(x)}{x(q-1)}, \quad \mathscr{G}_q P(x) = P(qx).$$

The q-orthogonal polynomials were treated in the thesis of Medem [20] (see also [21]); the peculiar q-Laguerre–Hahn class was introduced also by Medem [20], and developed in detail in the thesis of Foupouagnigni [10] (see also [13,14] as examples of this class). When $G=0$, (2) becomes linear and this correspond to the q-semi-classical situation. The q-classical and q-semi-classical orthogonal polynomials appear, beside the two aforementioned thesis, in Refs. [1,13,14,18,21].

We will from now denote orthogonal polynomials by OP and Laguerre–Hahn by LH.

We define the co-recursive [6] $(P_n^{[\mu]})_n$ of $(P_n)_n$ and the rth associated $(P_n^{(r)})_n$ of $(P_n)_n$ as the two families of monic polynomials defined by the following three terms recurrence relations obtained by modifying (1):

$$P_{n+1}^{[\mu]} = (x-\beta_n)P_n^{[\mu]} - \gamma_n P_{n-1}^{[\mu]}, \quad n \geqslant 1, \quad P_0^{[\mu]} = 1, \quad P_1^{[\mu]} = x - \beta_0 - \mu,$$

$$P_{n+1}^{(r)} = (x-\beta_{n+r})P_n^{(r)} - \gamma_{n+r} P_{n-1}^{(r)}, \quad n \geqslant 1, \quad P_0^{(r)} = 1, \quad P_1^{(r)} = x - \beta_r, \tag{3}$$

where μ is a complex number.

The co-recursive rth associated $(P_n^{\{r,\mu\}})_n$ of $(P_n)_n$ is defined as the co-recursive of the rth associated $(P_n^{(r)})_n$ of $(P_n)_n$. This family satisfies the relation

$$P_{n+1}^{\{r,\mu\}} = (x-\beta_{n+r})P_n^{\{r,\mu\}} - \gamma_{n+r} P_{n-1}^{\{r,\mu\}}, \quad n \geqslant 1, \quad P_0^{\{r,\mu\}} = 1, \quad P_1^{\{r,\mu\}} = x - \beta_r - \mu. \tag{4}$$

These families, by Favard theorem [7], are orthogonal. We denote by $\mathscr{U}^{[\mu]}$, $\mathscr{U}^{(r)}$ and $\mathscr{U}^{\{r,\mu\}}$, respectively, the regular normalized functionals associated with these OP families.

Obviously, we have the relations

$$P_n^{\{0,\mu\}} = P_n^{[\mu]}; \quad P_n^{\{r,0\}} = P_n^{(r)}.$$

The families $(P_n^{[\mu]})_n$, $(P_n^{(r)})_n$ and $(P_n^{\{r,\mu\}})_n$ belong, in general, to the LH class if $(P_n)_n$ belongs to the LH class [9,10,19,23,24]. As a consequence, any polynomial $P_n^{[\mu]}$, $P_n^{(r)}$ and $P_n^{\{r,\mu\}}$ satisfy a fourth-order differential or difference or q-difference equation with polynomial coefficients.

The fourth-order differential or difference equation satisfied by $P_n^{[\mu]}$ was given in [23,24] for classical continuous OP; and for classical discrete OP in [16]. This equation for q-classical OP was given in [12].

Differential, difference and q-difference equations satisfied by the rth associated OP of the LH class were given in details in [4,10,11,13,14].

On the other hand, the fourth-order differential or difference equation satisfied by the co-recursive associated OP was given in [15] for Laguerre and Jacobi OP; and for Meixner and Charlier OP in [16].

In this work, we first prove that the co-recursive rth associated $\mathscr{D}_q$-LH orthogonal polynomials is a $\mathscr{D}_q$-LH orthogonal polynomials; and use relations between the P_n, $P_n^{[\mu]}$, $P_n^{(r)}$, the co-recursive associated, $P_n^{\{r,\mu\}}$ of P_n and the q-difference equations satisfied by the associated OP of the D_q-LH class [10,13,11,14] to derive the fourth-order q-difference equation satisfied by the co-recursive rth associated OP of the D_q-Laguerre–Hahn class.

This q-difference equation is given explicitly for the co-recursive first associated classical OP and also for the co-recursive rth associated discrete q-Hermite II OP.

The q-Difference or differential equations obtained in the framework of this paper can be used to:

- Solve connection coefficients and linearization problems;
- To prove that the co-recursive rth associated of q-classical OP are of $\mathscr{D}_q$-Laguerre–Hahn class but *neither* q-classical *nor* q-semi-classical (for $\mu \neq 0$ and $r \geqslant 1$).

2. q-Difference equations for $P_n^{\{r,\mu\}}$

2.1. The co-recursive rth associated orthogonal polynomials

We state and prove the following lemma.

Lemma 1. *The co-recursive rth associated OP of the $\mathscr{D}_q$-LH class is of $\mathscr{D}_q$-LH class.*

Proof. Let $\mathscr{U}$ be a regular functional of the $\mathscr{D}_q$-LH class and let $(P_n)_n$ be the monic family orthogonal with respect to $\mathscr{U}$. Let S (resp. S_r and $S_{r,\mu}$) be the Stieltjes function of the functional $\mathscr{U}$ (resp. $\mathscr{U}^{(r)}$ and $\mathscr{U}^{\{r,\mu\}}$). It is well known [10,13] that when $\mathscr{U}$ is of $\mathscr{D}_q$-LH class, then $\mathscr{U}^{(r)}$ is of $\mathscr{D}_q$-LH. The Stieltjes function S_r of $\mathscr{U}^{(r)}$ therefore satisfies the Riccati q-difference equation:

$$\phi(qx)\mathscr{D}_q S_r(x) = G_r(x,q)S_r(x)\mathscr{G}_q S_r(x) + E_r(x,q)S_r(x) + F_r(x,q)\mathscr{G}_q S_r(x) + H_r(x,q), \tag{5}$$

where $\phi \neq 0$, G_r, E_r, F_r, G_r and H_r are polynomials. It shall be mentioned that $S = S_0$ satisfies

$$\phi(qx)\mathscr{D}_q S(x) = G_0(x,q)S(x)\mathscr{G}_q S(x) + E_0(x,q)S(x) + F_0(x,q)\mathscr{G}_q S(x) + H_0(x,q),$$

with the coefficients of the previous equation and those of (2) related by

$$G_0 = G, \quad E_0 = E, \quad F_0 = F, \quad H_0 = H.$$

We use the relation linking S_r and $S_{r,\mu} = S(\mathscr{U}^{\{r,\mu\}})$ [22]

$$S_{r,\mu} = \frac{S_r}{1 + \mu S_r}$$

and Eq. (5) to get the Riccati q-difference equation satisfied by $S_{r,\mu}$:

$$\phi(qx)\mathscr{D}_q S_{r,\mu}(x)$$
$$= G_{r,\mu}(x,q)S_{r,\mu}(x)\mathscr{G}_q S_{r,\mu}(x) + E_{r,\mu}(x,q)S_{r,\mu}(x) + F_{r,\mu}(x,q)\mathscr{G}_q S_{r,\mu}(x) + H_{r,\mu}(x,q), \tag{6}$$

with

$$G_{r,\mu} = G_r - \mu(E + F_r) + \mu^2 H_r, \quad E_{r,\mu} = E_r - \mu H_r, \quad F_{r,\mu} = F_r - \mu H_r, \quad H_{r,\mu} = H_r. \tag{7}$$

$\mathscr{U}^{\{r,\mu\}}$ is therefore of the $\mathscr{D}_q$-LH class. As a consequence, any polynomial $P_n^{\{r,\mu\}}$ satisfies a fourth-order linear q-difference equation with polynomial coefficients. □

2.2. The coupled equations for $P_n^{\{r,\mu\}}$

To derive these fourth-order q-difference equations, we first recall the following needed lemma giving the coupled equations linking the associated OP of the $\mathscr{D}_q$-LH class $P_n^{(r)}$ and $P_{n-1}^{(r+1)}$.

Lemma 2 (Foupouagnigni [10], Foupouagnigni et al. [13]). *If $(P_n)_n$ denotes the sequence of monic OP of the $\mathscr{D}_q$-LH class, then the rth associated $(P_n^{(r)})_n$ and $(P_{n-1}^{(r+1)})$ of $(P_n)_n$ satisfy*

$$D_{r,n}^q[P_n^{(r)}] = N_{r+1,n-1}^q[P_{n-1}^{(r+1)}], \tag{8}$$

$$D_{r+1,n-1}^q[P_{n-1}^{(r+1)}] = N_{r,n}^q[P_n^{(r)}], \tag{9}$$

where the q-difference operators are given by

$$D_{r,n}^q = a_2(r,n,x)\mathscr{G}_q^2 + a_1(r,n,x)\mathscr{G}_q + a_0(r,n,x), \quad N_{r+1,n-1}^q = \tilde{a}_1(r,n,x)\mathscr{G}_q + \tilde{a}_0(r,n,x),$$

$$D_{r+1,n-1}^q = b_2(r,n,x)\mathscr{G}_q^2 + b_1(r,n,x)\mathscr{G}_q + b_0(r,n,x), \quad N_{r,n}^q = \tilde{b}_1(r,n,x)\mathscr{G}_q + \tilde{b}_0(r,n,x).$$

The coefficients $a_j(r,n,x)$, $\tilde{a}_j(r,n,x)$, $b_j(r,n,x)$ and $\tilde{b}_j(r,n,x)$ are given by

$$\begin{aligned}
&a_2 = K_{3,0}(K_{1,1}K_{7,1} - K_{3,1}K_{8,1}), \quad a_1 = -K_{2,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}),\\
&a_0 = K_{3,1}(K_{2,0}K_{2,1} + K_{4,1}K_{6,0}),\\
&\tilde{a}_1 = K_{4,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}), \quad \tilde{a}_0 = -K_{3,1}(K_{2,1}K_{4,0} + K_{4,1}K_{5,0}),\\
&b_2 = K_{3,0}(K_{1,1}K_{7,1} - K_{3,1}K_{8,1}), \quad b_1 = -K_{5,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}),\\
&b_0 = K_{3,1}(K_{5,0}K_{5,1} + K_{4,0}K_{6,1}),\\
&\tilde{b}_1 = K_{6,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}), \quad \tilde{b}_0 = -K_{3,1}(K_{5,1}K_{6,0} + K_{6,1}K_{2,0})
\end{aligned} \tag{10}$$

with coefficients $K_{i,j}$ given by

$$K_{i,0}(r,n,q;x) \equiv K_i(r,n,q;x) \quad \text{and} \quad K_{i,j}(r,n,q;x) \equiv K_i(r,n,q;q^j x)$$

and

$$K_1(r,n,q;x) = \frac{\phi(qx)}{(q-1)x} + E_{n+r+1}(x,q), \quad K_2(r,n,q;x) = \frac{\phi(qx)}{(q-1)x} - F_r(x,q),$$

$$K_3(r,n,q;x) = \frac{H_{n+r}(x,q)}{\gamma_{n+r}}, \quad K_4(r,n,q;x) = \begin{cases} \gamma_r \frac{H_{r-1}(x,q)}{\gamma_{r-1}} & \text{if } r \geqslant 1, \\ \gamma_0 G_0 & \text{if } r = 0, \end{cases}$$

$$K_5(r,n,q;x) = \frac{\phi(qx)}{(q-1)x} + E_r(x,q), \quad K_6(r,n,q;x) = -\frac{H_r(x,q)}{\gamma_r},$$

$$K_7(r,n,q;x) = \frac{\phi(qx)}{(q-1)x} - F_{n+r+1}(x,q), \quad K_8(r,n,q;x) = -\gamma_{n+r+1}\frac{H_{n+r+1}(x,q)}{\gamma_{n+r+1}}. \tag{11}$$

Secondly, we state and prove the following proposition giving a link between the rth associated and the co-recursive rth associated OP.

Proposition 3. *Given a family of monic OP $(P_n)_n$, it's rth associated $P_n^{(r)}$ and its co-recursive rth associated $P_n^{\{r,\mu\}}$ satisfy the following relations*:

$$P_n^{\{r,\mu\}} = P_n^{(r)} - \mu P_{n-1}^{(r+1)}, \tag{12}$$

$$P_n^{\{r+1,\mu\}} = \frac{\mu}{\gamma_{r+1}} P_{n+1}^{(r)} + \left(1 - \frac{\mu(x-\beta_r)}{\gamma_{r+1}}\right) P_n^{(r+1)}, \tag{13}$$

where β_n and γ_n are the coefficients of the recurrence relation satisfied by $(P_n)_n$ (see (1)).

Proof. The proof is obtained using the fact that for any fixed x, the sequences $(P_n^{(r)}(x))_n$, $(P_{n-1}^{(r+1)}(x))_n$ and $(P_n^{\{r,\mu\}}(x))_n$ satisfy the same second-order difference equation (see (1)). Since the set of solutions of this second-order difference equations is a two-dimensional vector space containing $(P_n^{(r)}(x))_n$, $(P_{n-1}^{(r+1)}(x))_n$ and $(P_n^{\{r,\mu\}}(x))_n$, taking care of the fact that the families $(P_n^{(r)}(x))_n$ and $(P_{n-1}^{(r+1)}(x))_n$ are linearly independent, we deduce that there exists two constants $C_1(x)$, $C_2(x)$ such that

$$P_n^{\{r,\mu\}}(x) = C_1(x)P_n^{(r)}(x) + C_2(x)P_{n-1}^{(r+1)}(x), \quad n \geqslant 0.$$

Computations involving Eqs. (1), (3) and (4) gives $C_1(x) = 1, C_2(x) = -\mu$. It should be mentioned that relation (12) is given in [8] but only for the case where $r = 0$.

The same process applied to the families $(P_n^{(r+1)}(x))_n$, $(P_{n+1}^{(r)}(x))_n$ and $(P_n^{\{r+1,\mu\}}(x))_n)$ give, by analogy, relation (13). □

2.3. q-Difference equations for $P_n^{\{r,\mu\}}$

In the first step, we eliminate the term $P_{n-1}^{(r+1)}$ in Eqs. (8) and (9) using Eq. (12) and get the coupled relations linking the polynomials families $(P_n^{(r)})_n$ and $(P_n^{\{r,\mu\}})_n$, the two relations which used together lead us to the following proposition.

Proposition 4. *If $(P_n)_n$ denotes the sequence of monic OP of the $\mathscr{D}_q$-LH class; then the associates $(P_n^{(r)})_n$ and the co-recursive associated $(P_n^{\{r,\mu\}})$ of $(P_n)_n$ satisfy*

$$D_{r,n}^{q,\mu}[P_n^{\{r,\mu\}}] = N_{r,n}^{q,\mu}[P_n^{(r)}], \tag{14}$$

$$\bar{D}_{r,n}^{q,\mu}[P_n^{(r)}] = \bar{N}_{r,n}^{q,\mu}[P_n^{\{r,\mu\}}], \tag{15}$$

where the q-difference operators are given by

$$D_{r,n}^{q,\mu} = c_2(r,n,x)\mathscr{G}_q^2 + c_1(r,n,x)\mathscr{G}_q + c_0(r,n,x), \quad N_{r,n}^{q,\mu} = \tilde{c}_1(r,n,x)\mathscr{G}_q + \tilde{c}_0(r,n,x),$$
$$\bar{D}_{r,n}^{q,\mu} = d_2(r,n,x)\mathscr{G}_q^2 + d_1(r,n,x)\mathscr{G}_q + d_0(r,n,x), \quad \bar{N}_{r,n}^{q,\mu} = \tilde{d}_1(r,n,x)\mathscr{G}_q + \tilde{d}_0(r,n,x), \tag{16}$$

with

$$c_j(r,n,x) \equiv c_j(x), \quad d_j(r,n,x) \equiv d_j(x), \quad \tilde{c}_j(r,n,x) \equiv \tilde{c}_j(x), \quad \tilde{d}_j(r,n,x) \equiv \tilde{d}_j(x)$$

and

$$c_2(x) = -b_2(x)a_2(x)\mu, \quad c_1(x) = -b_2(x)\tilde{a}_1(x) - a_2(x)\mu b_1(x),$$
$$c_0(x) = -b_2(x)\tilde{a}_0(x) - b_0(x)a_2(x)\mu,$$
$$\tilde{c}_1(x) = -a_2(x)\mu b_1(x) - b_2(x)\tilde{a}_1(x) + b_2(x)a_1(x)\mu + a_2(x)\mu^2\tilde{b}_1(x),$$
$$\tilde{c}_0(x) = -b_2(x)\tilde{a}_0(x) + a_2(x)\mu^2\tilde{b}_0(x) + b_2(x)a_0(x)\mu - b_0(x)a_2(x)\mu,$$
$$d_2(x) = a_2(x)\mu, \quad d_1(x) = -\tilde{a}_1(x) + a_1(x)\mu, \quad d_0(x) = -\tilde{a}_0(x) + a_0(x)\mu,$$
$$\tilde{d}_1(x) = -\tilde{a}_1(x), \quad \tilde{d}_0(x) = -\tilde{a}_0(x). \tag{17}$$

The coupled equations linking $(P_n^{(r)})_n$ and $(P_n^{\{r,\mu\}})$ are essential equations for the derivation of the fourth-order q-difference equations satisfied by the co-recursive associated OP of the $\mathscr{D}_q$-LH class.

In the first step we apply the operator $\mathscr{G}_q$ to Eq. (14) and use Eq. (15) to eliminate the term $P_n^{(r)}(q^2x)$ and get

$$(e_3(r,n,x)\mathscr{G}_q^3 + e_2(r,n,x)\mathscr{G}_q^2 + e_1(r,n,x)\mathscr{G}_q + e_0(r,n,x))P_n^{\{r,\mu\}}$$
$$=(\tilde{e}_1(r,n,x)\mathscr{G}_q + \tilde{e}_0(r,n,x))P_n^{(r)} \tag{18}$$

with coefficients e_j, and $\tilde{e}_j$ given, by $e_j(r,n,x) \equiv e_j(x)$, $\tilde{e}_j(r,n,x) \equiv \tilde{e}_j(x)$ and

$$e_3(x) = c_2(xq)d_2(x), \quad e_2(x) = c_1(xq)d_2(x), \quad e_1(x) = c_0(xq)d_2(x) - \tilde{c}_1(xq)\tilde{d}_1(x),$$
$$e_0(x) = -\tilde{c}_1(xq)\tilde{d}_0(x), \quad \tilde{e}_1(x) = \tilde{c}_1(xq)d_1(x) - \tilde{c}_0(xq)d_2(x), \quad \tilde{e}_0(x) = \tilde{c}_1(xq)d_0(x). \tag{19}$$

In the second step, we apply the operator $\mathscr{G}_q$ to Eq. (18) and use again Eq. (15) to eliminate $P_n^{(r)}(q^2x)$ to get

$$(f_4(r,n,x)\mathscr{G}_q^4 + f_3(r,n,x)\mathscr{G}_q^3 + f_2(r,n,x)\mathscr{G}_q^2 + f_1(r,n,x)\mathscr{G}_q + f_0(r,n,x))P_n^{\{r,\mu\}}$$
$$=(\tilde{f}_1(r,n,x)\mathscr{G}_q + \tilde{f}_0(r,n,x))P_n^{(r)} \tag{20}$$

with coefficients f_j, and $\tilde{f}_j$ given by $f_j(r,n,x) \equiv f_j(x)$; $\tilde{f}_j(r,n,x) \equiv \tilde{f}_j(x)$ and

$$f_4(x) = c_2(xq^2)d_2(xq)d_2(x), \quad f_3(x) = c_2(xq)d_2(x),$$
$$f_2(x) = d_2(x)(c_0(xq^2)d_2(xq) - \tilde{c}_1(xq^2)\tilde{d}_1(xq)),$$
$$f_1(x) = -\tilde{c}_1(xq^2)\tilde{d}_0(xq)d_2(x) + \tilde{c}_1(xq^2)d_1(xq)\tilde{d}_1(x) - \tilde{c}_0(xq^2)d_2(xq)\tilde{d}_1(x),$$

$$f_0(x) = -\tilde{d}_0(x)(-\tilde{c}_1(xq^2)d_1(xq) + \tilde{c}_0(xq^2)d_2(xq)),$$

$$\tilde{f}_1(x) = -\tilde{c}_1(xq^2)d_1(xq)d_1(x) + \tilde{c}_1(xq^2)d_0(xq)d_2(x) + \tilde{c}_0(xq^2)d_2(xq)d_1(x),$$

$$\tilde{f}_0(x) = d_0(x)(-\tilde{c}_1(xq^2)d_1(xq) + \tilde{c}_0(xq^2)d_2(xq)). \tag{21}$$

From elimination of the right-hand side of Eqs. (14), (18) and (20), we see that $P_n^{\{r,\mu\}}$ satisfies a fourth-order q-difference equation (with polynomials coefficients) easily written in 3×3 determinant. This equation can be written in terms of the operator $\mathscr{G}_q$ as

$$(I_4(r,n,\mu,x)\mathscr{G}_q^4 + I_3(r,n,\mu,x)\mathscr{G}_q^3 + I_2(r,n,\mu,x)\mathscr{G}_q^2 + I_1(r,n,\mu,x)\mathscr{G}_q + I_0(r,n,\mu,x))P_n^{\{r,\mu\}} = 0, \tag{22}$$

where I_k are polynomials with fixed degrees.

3. Applications

3.1. Co-recursive rth associated q-classical orthogonal polynomials

- We suppose that the regular functional $\mathscr{U}$ is represented by the q-classical weight ρ (defined on the set I) satisfying the equation

$$\mathscr{D}_q(\phi\rho) = \psi\rho, \tag{23}$$

where ϕ is a polynomial of degree at most two and ψ a first-degree polynomial.

Then we deduce that $\mathscr{U}$ is q-classical (see [1,20,21]) and satisfies the functional equation

$$\mathscr{D}_q(\phi\mathscr{U}) = \psi\mathscr{U}. \tag{24}$$

The coefficients $\beta_r, \gamma_r, G_r,\ E_r, F_r, G_r$ and H_r (see Eq. (5)) in this case are given explicitly in [10,13,20].

The coefficients $E_{r,\mu}, F_{r,\mu}, G_{r,\mu}$, and $H_{r,\mu}$ are computed using Eqs. (7) and the expressions of $G_r,\ E_r,\ F_r,\ G_r$ and H_r given in [10,13]. In particular, $G_{r,\mu}$ is given by

$$G_{r,\mu}(x,q) = -\mu\left(q^r\psi_1 + \frac{q^r - q^{2-r}}{q-1}\phi_2\right)x + A(r,q), \tag{25}$$

where the constant $A(r,q)$ is space consuming.

- Since $\psi_1 \neq 0$, the coefficients $G_{r,\mu}$ is different from zero for $r \geqslant 1$ and $\mu \neq 0$. This result permits us to conclude that the co-recursive rth associated classical OP is neither classical nor semi-classical.
- Since the coefficients $I_j(r,n,x)$ are too large, we are going to give, for illustration, coefficients a_j, $\tilde{a}_j,\ b_j$ and $\tilde{b}_j$ for Discrete q-Hermite II case. One can then deduce coefficients $I_j(r,n,\mu,x)$ since they are written in terms of $a_j,\ \tilde{a}_j,\ b_j$ and $\tilde{b}_j$.

The polynomials coefficients a_j, $\tilde{a}_j$, b_j and $\tilde{b}_j$ of Eqs. (8) and (9) for the discrete q-Hermite II case ($\phi(x)=1$; $\psi(x)=x/(1-q)$) are given by

$$
\begin{aligned}
&\mathrm{a}_2(x)=(-1+qx)(qx+1)q^nq^r, \quad \mathrm{a}_1(x)=-q^nq^r(-1-q+q^rq^nq^2x^2),\\
&\mathrm{a}_0(x)=q(-1+qx^2q^r-qx^2)q^nq^r, \quad \tilde{\mathrm{a}}_1(x)=-xq(-1-q+q^rq^nq^2x^2)q^n(q^r-1),\\
&\tilde{\mathrm{a}}_0(x)=xq(q^r-1)(qx^2q^r-1-q)q^n,\\
&\mathrm{b}_2(x)=(-1+qx)(qx+1)q^nq^r, \quad \mathrm{b}_1(x)=(-1-q+q^rq^nq^2x^2)q^rq^n(-1+q^rq^2x^2),\\
&\mathrm{b}_0(x)=(-1-q+q^rq^nq^2x^2)q^rq^n(-1+q^rq^2x^2),\\
&\tilde{\mathrm{b}}_1(x)=xq(-1-q+q^rq^nq^2x^2)q^n(q^r)^2, \quad \tilde{\mathrm{b}}_0(x)=-xq(-1+q^rq^2x^2-q)q^n(q^r)^2.
\end{aligned}
\tag{26}
$$

3.2. Particular cases

Uses of Eqs. (8), (9) and (12)–(15) permit us to recover known results [10,13,12]: The fourth-order q-difference equations satisfied by the rth associated orthogonal polynomials of the D_q-LH class, as well as the fourth-order q-difference equations satisfied by the co-recursive orthogonal polynomials of the D_q-LH.

3.3. Co-recursive first associated q-classical orthogonal polynomials

When we set $r=1$ in Eq. (22), we get the fourth-order q-difference equation satisfied by the co-recursive OP of the $\mathscr{D}_q$-LH class.This equation, when the initial family $(P_n)_n$ is q-semi-classical, can be factorized as product of two second-order q-difference equation. In fact, we eliminate the term $P_{n-1}^{(r+1)}$ in Eqs. (8) and (9) using relation (13) (instead of (12)); and get the two coupled equations linking $P_{n-1}^{\{r+1,\mu\}}$ and $P_n^{(r)}$. These equations for $r=0$ reach as

$$\tilde{D}_{0,n}^{q,\mu}[P_{n-1}^{\{1,\mu\}}]=\tilde{N}_{0,n}^{q,\mu}[P_n], \tag{27}$$

$$\tilde{D}_{0,n}^{q,\mu}[P_n]=\tilde{N}_{0,n}^{q,\mu}[P_{n-1}^{\{1,\mu\}}]. \tag{28}$$

The second-order linear q-difference operator, $\tilde{D}_{0,n}^{*q,\mu}$, annihilating the second hand of (27) when $(P_n)_n$ is q-semi-classical, is obtained thanks to the second-order linear q-difference equation satisfied by the q-semi-classical OP $(P_n)_n$. The fourth-order difference equation reaches as

$$\tilde{D}_{0,n}^{*q,\mu}\tilde{D}_{0,n}^{q,\mu}(P_{n-1}^{\{1,\mu\}})=0.$$

In particular, we have the following:

The co-recursive first associated $P_{n-1}^{\{1,\mu\}}$ of the q-classical OP $(P_n)_n$ orthogonal with respect to the q-classical weight ρ satisfying $\mathscr{D}_q(\phi\rho)=\psi\rho$, where ϕ is a polynomial of degree at most two and ψ a first-degree polynomial satisfies:

$$\tilde{D}_{0,n}^{q,\mu}[P_{n-1}^{\{1,\mu\}}]=\tilde{N}_{0,n}^{q,\mu}[P_n], \tag{29}$$

where

$$\tilde{D}^{q,\mu}_{0,n} = (\phi_{(1)} + \psi_{(1)}t_{(1)})\mathcal{Q}^{\{1,\mu\}}_{2,n-1}, \quad \tilde{N}^{q,\mu}_{0,n} = \tilde{g}\mathcal{G}_q + \tilde{h}\mathcal{I}_d \tag{30}$$

with

$$\begin{aligned}
\mathcal{Q}^{\{1,\mu\}}_{2,n-1} &= MM_{(1)}\phi_{(2)}\gamma_1\mathcal{G}_q^2 - MM_{(2)}((1+q)\phi_{(1)} + \psi t_{(1)} - \lambda_{q,n}t_{(1)}^2)\gamma_1\mathcal{G}_q \\
&\quad + M_{(1)}M_{(2)}q(\phi + \psi t)\gamma_1\mathcal{I}_d, \\
\tilde{g} &= (\mu M_{(1)}\phi_{(2)} - \mu M_{(2)}\phi_{(1)} - \mu M_{(2)}\psi_{(1)}t_{(1)} + M_{(1)}M_{(2)}ct_{(1)}\gamma_1) \\
&\quad \times((1+q)\phi_{(1)} + \psi_{(1)}t_{(1)} - \lambda_{q,n}t_{(1)}^2)M, \\
\tilde{h} &= -(\mu qM\phi_{(2)}\phi_{(1)} - \mu qM_{(2)}\phi\phi_{(1)} - \mu qM_{(2)}\phi\psi_{(1)}t_{(1)} \\
&\quad - \mu qM_{(2)}\psi t\phi_{(1)} - \mu qM_{(2)}\psi t\psi_{(1)}t_{(1)} \\
&\quad + MM_{(2)}ct_{(1)}\gamma_1\phi_{(1)} + MM_{(2)}ct_{(1)}\gamma_1\phi_{(1)}q + MM_{(2)}ct_{(1)}^2\gamma_1\psi_{(1)})M_{(1)}.
\end{aligned} \tag{31}$$

where

$M_{(j)} \equiv M(q^jx)$, $M_{(0)} \equiv M(x) = 1-(\mu(x-\beta_0))/\gamma_1$, $c = \phi''/2 - \psi'$, $\beta_0 = -\psi_0/\psi_1$, $\gamma_1 = -\phi(\beta_0)/(\phi_2 + q\psi_1)$,
$\phi_{(j)} = \phi(q^jx)$, $\psi_{(t)} = \psi(q^jx)$, $t_{(j)} = t(q^jx)$, $t(x) = (q-1)x$,
$\lambda_{q,n} = -[n]_q\{\psi' + [n-1]_{1/q}\phi''/2q\}$, $[n]_q = (1-q^n)/(1-q)$.

The factorized form of the fourth-order q-difference equation satisfied by the co-recursive first associated q-classical OP

$$\mathcal{Q}^{*\{1,\mu\}}_{2,n-1}\mathcal{Q}^{\{1,\mu\}}_{2,n-1}[P^{\{1,\mu\}}_{n-1}(x)] = 0$$

is obtained using (29) and the second-order q-difference equation satisfied by P_n:

$$(\phi(x)\mathcal{D}_q\mathcal{D}_{1/q} + \psi(x)\mathcal{D}_q + \lambda_{q,n})P_n = 0.$$

In fact, $\mathcal{Q}^{*\{1,\mu\}}_{2,n-1}$ which is a second-order linear q-difference equation with polynomial coefficients is obtained by applying twice the operator $\mathcal{G}_q$ to Eq. (29) and using the previous equation to eliminate the term $P_n(q^2x)$. Since $\mathcal{Q}^{*\{1,\mu\}}_{2,n-1}$ is space consuming, we decide to give it for the discrete q-Hermite II case. The operators $\mathcal{Q}^{\{1,\mu\}}_{2,n-1}$ and $\mathcal{Q}^{*\{1,\mu\}}_{2,n-1}$ of the co-recursive first associated of the discrete q-Hermite II OP (using Maple V [5] for computations) are given by

$$\begin{aligned}
\mathcal{Q}^{\{1,\mu\}}_{2,n-1} &= -(-q+1+\mu q^2x)(-q+1+\mu qx)\mathcal{G}_q^2 \\
&\quad - (-q+1+\mu q^3x)(-q+1+\mu qx)(q^2x^2q^n - 1 - q)\mathcal{G}_q \\
&\quad + q(x-1)(x+1)(-q+1+\mu q^3x)(-q+1+\mu q^2x), \\
\mathcal{Q}^{*\{1,\mu\}}_{2,n-1} &= (q^2x-1)(q^2x+1)(qx-1)(qx+1)(q^4x^2q^n - 1 - q - q^2 - q^3 + q^4x^2)^2\mathcal{G}_q^2 \\
&\quad - q^2(qx-1)(qx+1)(q^4x^2q^n - 1 - q - q^2 - q^3 + q^4x^2)(q^9x^4(q^n)^2 + q^9x^4q^n \\
&\quad - q^8x^2q^n - q^8x^2 - q^7x^2q^n - q^7x^2 - q^6x^2q^n - q^5x^2q^n - q^4x^2q^n - q^3x^2q^n
\end{aligned}$$

$$+q^4+2q^3+2q^2+2q+1)\mathscr{G}_q - q^5(q^2x-1)(q^2x+1)$$
$$(q^6x^2q^n+q^6x^2-q^3-q^2-q-1)(q^4x^2q^n-1-q-q^2-q^3+q^4x^2).$$

3.4. Concluding remarks

- If for q-classical OP, there are conditions under which associated and co-recursive of classical OP are still classical [10,14], this is *not* the case for the co-recursive rth associated q-classical OP (with $\mu \neq 0$, $r \geqslant 1$).
- Using the results obtained in the framework of this paper, we have deduced the fourth-order differential equation satisfied by the co-recursive rth associated LH orthogonal polynomials, as well as the fourth-order difference equation satisfied by the co-recursive rth associated OP of the Δ-LH class. This allows us to give the coefficients of the fourth-order differential and difference equation for the co-recursive rth associated Bessel, Hermite, Hahn and Krawtchouk OP, results which seem to be new and extend some results given by Letessier.
- The q-Difference equations given in this paper can be used to solve linearization problem like in [3] and even in [2] when the expanding family is not orthogonal (q-Pochhammer), and also to construct the recurrence relation for the connection coefficients of Fourier coefficients as done in [17,18].

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